Burgers type equations, Gelfand's problem and Schumpeterian dynamics

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Abstract. Burgers equations have been introduced to study different models of fluids Bateman, 1915, Burgers, 1939, Hopf, 1950, Cole, 1951, Lighthill, Whitham, 1955.... The difference-differential analogs of these equations have been proposed for Schumpeterian models of economic development Iwai, 1984, Polterovich, Henkin, 1988, Belenky, 1990, 1996, Henkin, Polterovich, 1999, Shananin, Tashlitskaya, 2000....

This paper is a survey of recent results and conjectures on Burgers type equations, motivated both by fluid mechanics and by Schumpeterian dynamics. Abriged proofs of new results are given. This paper is an extended version of the paper [H2] prepared for the talk at the conference "General Equilibrium Analysis" at Higher School of Economics, June, 2011.

1. Introduction: Burgers type equations and Schumpeterian dynamics.

By Burgers type equations we mean scalar partial differential equations of the form

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \varepsilon\frac{\partial^2 f}{\partial x^2}, \quad \varepsilon > 0, \tag{1a}$$

$f = f(x,t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, and the scalar difference-differential equations of the form

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{f(x,t) - f(x-\varepsilon,t)}{\varepsilon} = 0, \quad x = k\varepsilon, \ k \in \mathbb{Z} \tag{1b}$$

and also variations and multidimensional generalization of (1a), (1b).

For linear $f \mapsto \varphi(f)$ equation (1a) was introduced by H.Bateman, 1915, and deeply studed by J.Burgers, 1939, as a simplest model for pressure-less gas dynamics. For general $\varphi(f)$ equation (1a) has appeared later in very different models, for example: in the model for displacement of oil by water (S.Buckley, M.Leverett, 1942), in the model of consolidation of wet soil (Florin, 1948) , in the model of the road traffic (M.Lighthill, G.Whitham, 1955) etc.

The equation (1b) was proposed in Polterovich, Henkin, 1988,1989 for the description of a Schumpeterian evolution of industry.

According to Schumpeter 1911, 1939, economical development of industry is periodical process with period of order half-century ("business cycle"). It consists in cascades of creation, processes of formation and cascades of destruction. Creative and destructive cascades can be described by Lotka-Volterra type equations (P.Hanel, P.Klimek, S.Thurner, 2008, 2010).

The mechanism of technological changes in the industry during processes of formation can be divided into two components: creation of new technologies by a firm (innovation process) and adoption of technologies, created by other firms (imitation process). For an industry with many firms its development can be described as evolution of its efficiency distribution. Let $f_n(t)$ be a share of firms at the moment t in the given industry that have efficiency level $\leq n$.

The simplest model of Schumpeterian dynamics of industry has the form

$$\frac{df_n}{dt} = \varphi(f_n)(f_{n-1}(t) - f_n(t)), \quad \text{where} \quad \varphi(f_n) = \alpha + \beta(1 - f_n),$$

$\alpha \geq 0$ be the share of firms moving from level n to the level $(n+1)$ per unit of time due to innovation, $\beta(1-f_n)$ be the share of firms moving from level n to the level $(n+1)$ per unit of time due to imitation.

In applications this model may appear in different forms.

In the model with depreciation of capacities (Polterovich, Henkin, 1988) the following modification of the simplest model has appeared:

$$\frac{df_n}{dt} = \varphi(f_n)(f_{n-1}(t) - f_n(t)) + \mu(f_{n+1}(t) - f_n(t)), \qquad (*)$$

where μ is depreciation rate.

This model was succesfully implemented numerically (Gelman, Levin, Polterovich, Spivak, 1993) for description of evolution of distribution by efficiency levels for the Ferrous Metallurgy in USSR. Using automodel reduction $f_n(t) = f(t, nh)$ with $h \to 0$, A.Gasnikov and A.Shananin (2006) have found hypothetique formula for the velocity of travelling wave propagation for (*).

In the model of economics growth (Polterovich, Henkin, 1989) we put $f_n = \sum\limits_{k=0}^{n} M_k / \sum\limits_{k=0}^{\infty} M_k$, where M_n are capacities of the level n, and we have

$$\frac{dM_n(t)}{dt} = (1 - \varphi_0(f_n))\lambda_n M_n + \varphi_0(f_{n-1})\lambda_{n-1} M_{n-1}.$$

Here λ_n - profit per unit of capacities per unit of time, $\varphi_0(f_n)$- the share of profit $\lambda_n M_n$, creating new capacities of the level $n+1$.

If $\lambda_k \uparrow \lambda > 0$ and $\sum\limits_{k=1}^{\infty} k(\lambda - \lambda_k) < \infty$, this gives the following variation of the simplest model

$$\frac{df_n}{dt} = \varphi(f_n)(f_{n-1} - f_n) + r_n,$$

where $\varphi = \lambda\varphi_0$ and r_n is a term nonessential for asymptotic behaviour.

In the model of Belenky, 1996, the speed of transition from efficiency level n to level $(n+1)$ depends on a proportion $r_n = (1-f_n)(1-f_{n-1})^{-1}$ of more advanced firms among all firms that are not worse than the firms of level n. This gives an interesting alternative for equation (1b):

$$\frac{d}{dt} \ln r_n(t) = \psi(r_n) - \psi(r_{n-1}),$$

where ψ is "motivation function", similar to function φ in (1b).

For physical applications of equation (1a) the main interest has the inviscid case of (1a), when $\varepsilon = +0$. But for transport flow models and for some social and biological applications the significant interest has the equation (1b) with $\varepsilon = 1$ and $x \in \mathbb{Z}$.

The results of finite-difference approximations for nonlinear conservation laws (see A.Harten, J.Hyman, P.Lax, 1976, Engquist, Osher, 1981, Henkin, Shananin, 2004) explain both the similarity of behavior of (1a), (1b) and also some difference in behavior of (1a) and (1b).

Motivated by models of fluid mechanics, Gel'fand (1959) has formulated following problem.

To find asymptotic ($t \to \infty$) of the solution $f(x,t)$ of the equation (1a) with initial condition

$$f(x,0) = \begin{cases} \alpha^{\pm}, & \text{if } \ \pm x > \pm x^{\pm} \\ f^0(x), & \text{if } \ x \in [x^-, x^+], \end{cases} \tag{2}$$

where $\alpha^- \leq \alpha^+$, $x^- \leq x^+$ and f^0 is a bounded function on $[x^-, x^+]$.

Gelfand (1959) has found a solution of this problem for the inviscid case $\varepsilon = +0$ with special initial conditions $f(x,0) = \alpha^{\pm}$, if $\pm x > 0$, and has noted:

it would be interesting to prove that the main term of asymptotic ($t \to \infty$) of $f(x,t)$ satisfying (1a), (2) coincides with the solution of (1a), (2) with $\varepsilon = +0$.

Motivated by models of economical development similar problems were considered later for equation (1b) in Henkin, Polterovich (1991, 1994, 1999). For formulations of precise conjectures and results, concerning equations (1a) and (1b) we will use assumptions and definitions going back to Gelfand.

2. Gelfand's problem. Main results.

In the asymptotics statements below we will not indicate as a rule the dependence of some constants from initial function $f^0(x)$ and sometimes from state function $\varphi(f)$.

Assumption 1.

Let $\alpha^- < \alpha^+$ and φ be a positive continuous differentiable function on the interval $[\alpha^-, \alpha^+]$ and φ' has only isolated zeros.

Put

$$\psi(u) = -\int_{\alpha^-}^{u} \varphi(y)dy, \quad u \in [\alpha^-, \alpha^+], \quad \text{for } \ (1.a), \tag{3a}$$

$$\psi(u) = \int_{\alpha^-}^{u} \frac{dy}{\varphi(y)}, \quad u \in [\alpha^-, \alpha^+], \quad \text{for } \ (1.b). \tag{3b}$$

Let us introduce respectively for (3a) and for (3b) the concave function $\hat{\psi}(u)$ as the upper bound of the convex hull of the set

$$\{(u,v): \ v \leq \psi(u), \ \ u \in [\alpha^-, \alpha^+]\}.$$

Assumption 2. For (3a) and (3b) the set $S = \{u \in [\alpha^-, \alpha^+] : \ \psi(u) < \hat{\psi}(u)\}$ has the following form

$$
\begin{aligned}
&S = (\alpha_0^-, \alpha_0^+) \cup (\alpha_1^-, \alpha_1^+) \cup \ldots (\alpha_L^-, \alpha_L^+), \quad \text{where} \\
&\alpha^- = \alpha_0^- \leq \alpha_0^+ < \alpha_1^- < \alpha_1^+ < \ldots < \alpha_{L-1}^- < \alpha_{L-1}^+ < \alpha_L^- \leq \alpha_L^+ = \alpha^+.
\end{aligned} \tag{4a, b}
$$

Let

$$
c_l = \frac{1}{\alpha_l^+ - \alpha_l^-} \int\limits_{\alpha_l^-}^{\alpha_l^+} \varphi(y)dy, \quad \text{for} \quad (1a), \quad l = 0, \ldots, L, \tag{5a}
$$

$$
c_l = (\alpha_l^+ - \alpha_l^-) \Big(\int\limits_{\alpha_l^-}^{\alpha_l^+} \frac{dy}{\varphi(y)} \Big)^{-1}, \quad \text{for} \quad (1b), \quad l = 0, \ldots, L. \tag{5b}
$$

Assumptions 1,2 imply (Gelfand, 1959, Oleinik, 1959, Weinberger, 1990, Henkin, Polterovich, 1999) the following important inequalities (for (1a)) and respectively for (1b):

$$
\begin{aligned}
&\varphi(\alpha_l^+) \leq c_l \leq \varphi(\alpha_l^-), \quad l = 0, \ldots, L, \\
&c_l = \varphi(\alpha_l^-), \quad l = 1, \ldots, L, \\
&c_l = \varphi(\alpha_l^+), \quad l = 0, \ldots, L-1.
\end{aligned} \tag{6}
$$

Let us remark that the inequalities above are, in fact, equalities except for the cases $l = 0$ and $l = L$.

Assumption 3. Let for (1a) and respectively for (1b) the following inequalities are valid

$$
\begin{aligned}
&\varphi'(\alpha_l^-) \neq 0, \quad l = 1, \ldots, L, \\
&\varphi'(\alpha_l^+) \neq 0, \quad l = 0, 1, \ldots, L-1, \\
&\varphi(\alpha_0^-) \neq c_0, \quad \text{if} \quad \alpha_0^- < \alpha_0^+, \\
&\varphi(\alpha_L^+) \neq c_L, \quad \text{if} \quad \alpha_L^- < \alpha_L^+.
\end{aligned}
$$

Theorem 1. (Generalized Gelfand theorem).

Let under assumptions 1-3 we have $\varepsilon = +0$. Let, in addition, initial data function $f^0(x)$ be the function of bounded variation on $[x^-, x^+]$. Then solutions of the Cauchy problems (1a,b), (2) have the following asymptotic structure

$$
f(x,t) \overset{L^1(\mathbb{R})}{\Rightarrow} \begin{cases} \alpha^-, & \text{if } x < c_0 t + d_0 \\ \varphi^{(-1)}(x/t), & \text{if } c_l t + d_l \leq x < c_{l+1} t + d_{l+1}, \ l = 0, 1, \ldots, L-1 \\ \alpha^+, & \text{if } x \geq c_L t + d_L, \end{cases}
$$

where $t \to \infty$, parameters $\{c_l\}$ determined by (5a,b) and parameters $\{d_l\}$ determined by corresponding equation (1a,b) and initial data (2), inverse function $\varphi^{(-1)}(\cdot)$ is well defined on $[c_l, c_{l+1}]$, $0 \leq l \leq L-1$.

- For equation (1a) with special initial condition $f(x,0)=\alpha^{\pm}$, if $\pm x>0$ Theorem 1 was obtained by Gelfand, 1959. Theorem 1 under condition $\alpha^{-}=\alpha_0^{-}<\alpha_0^{+}=\alpha^{+}$ was proved by Iljin, Oleinik, 1960, for equation (1a) and can be deduced from Henkin, Polterovich, 1991 for equation (1b).

- Under additional assumption that $f(x,t)$ is piecewise smooth function of $(x,t)\in\mathbb{R}\times\mathbb{R}^{+}$ and $f^0(x)\in[\alpha^{-},\alpha^{+}]$, $x\in[x^{-},x^{+}]$, Theorem 1 was obtained by Kruzhkov, Petrosyan, 1983, 87.

- Theorem 1 for equation (1a) only under assumption 1 was announced by Gasnikov, 2009.

- Theorem 1 for both equations (1a) and (1b) is deduced here by vanishing viscocity method from results of [H1] (see Theorems 2, 3 below). It gives the answer to the question formulated in [KP] ("...the ways of application of vanishing viscocity methods in this case are absolutely unknown...").

For formulation of main results (Theorems 2, 3) we need the following important statement about travelling waves solutions of equations (1a,b) belonging to Gelfand, 1959, Oleinik, 1959, for equation (1a) and Polterovich, Henkin, 1988, Belenky, 1990, for equation (1b).

Proposition 1.

Under assumptions 1,2 for $l\in\{0,\dots,L\}$ there exist (and unique up to translations) travelling wave solutions of (1a) and (1b) of the form

$$\begin{aligned}
&f=\tilde{f}_l\Big(\frac{x-c_lt}{\varepsilon}\Big) \text{ such that } \ \forall\varepsilon>0\\
&\tilde{f}_l\Big(\frac{x}{\varepsilon}\Big)\to\alpha_l^{\pm} \ \text{ as } \ x\to\pm\infty,\ l=0,\dots,L, \ \text{ and}\\
&\tilde{f}_l\Big(\frac{x}{\varepsilon}\Big)\to\alpha_l^{\pm} \ \text{ as } \ \varepsilon\to0, \ \text{ if } \ \pm x>0.
\end{aligned}$$

Theorem 2.

Let $\varepsilon\in(0,1)$. Under the assumptions 1,2,3 and definitions (3a,b), (4a,b), (5a,b) the solutions $f(x,t)$ of the Cauchy problems (1a,b), (2) have for $t\geq\varepsilon O\big(|\ln\frac{1}{\varepsilon}|^2\big)$ the following asymptotic structure:

$$f(x,t)=O\Big(\frac{\varepsilon}{t}\Big)^{1/4}+
\begin{cases}
\tilde{f}_l\big(\frac{1}{\varepsilon}(x-c_lt-d_l\big(\frac{t}{\varepsilon},\big(\frac{\varepsilon}{t}\big)^{1/4}\big))\big), & \text{if } |x-c_lt|\leq\sqrt{\varepsilon}\cdot(\varepsilon t)^{1/4},\quad l=0,1,\dots,L,\\
\varphi^{(-1)}\big(\frac{x}{t}\big), & \text{if } \ c_lt+\sqrt{\varepsilon}(\varepsilon t)^{1/4}\leq x\leq c_{l+1}t-\sqrt{\varepsilon}(\varepsilon t)^{1/4},\\
 & l=0,1,\dots,L-1,\\
\alpha^{-}, & \text{if } x\leq c_0t-\sqrt{\varepsilon}(\varepsilon t)^{1/4},\\
\alpha^{+}, & \text{if } x\geq c_Lt+\sqrt{\varepsilon}(\varepsilon t)^{1/4},
\end{cases} \tag{7}$$

where shift functions $d_l\big(\frac{t}{\varepsilon},A\big)$, $A=\big(\frac{\varepsilon}{t}\big)^{1/4}$ are determined by the "localized conservation laws" (see [HS])

$$\int\limits_{c_l t - A\sqrt{\varepsilon t}}^{c_l t + A\sqrt{\varepsilon t}} [f(x,t) - \tilde{f}_l\big(\frac{1}{\varepsilon}(x - c_l t - d_l\big(\frac{t}{\varepsilon}, A\big))\big)]dx = 0 \qquad (8a)$$

for equation (1a) and

$$\begin{aligned}&\sum_{k=[c_l t - A\sqrt{\varepsilon t}]+1}^{[c_l t + A\sqrt{\varepsilon t}]-1} (\Phi(f(k\varepsilon, t)) - \Phi\big(\tilde{f}_l\big(\frac{1}{\varepsilon}(k\varepsilon - c_l t - d_l\big(\frac{t}{\varepsilon}, A\big))\big)\big)\pm \qquad (8b)\\ &(c_l t \pm A\sqrt{\varepsilon t} - [c_l t \pm A\sqrt{\varepsilon t}])\times\\ &(\Phi(f([c_l t \pm A\sqrt{\varepsilon t}], t)) - \Phi(\tilde{f}_l([c_l t \pm A\sqrt{\varepsilon t}] - c_l t - d_l\big(\frac{t}{\varepsilon}, A\big)))\big),\end{aligned}$$

where $\Phi(f) = \int\limits_f^{\alpha^+} \frac{dy}{\varphi(y)}$ for equation (1b).

Theorem 3.

Under conditions of Theorem 2 the shift functions $d_l\big(\frac{t}{\varepsilon}, \big(\frac{\varepsilon}{t}\big)^{1/4}\big)$ have for $t \geq \varepsilon O\big(\ln\frac{1}{\varepsilon}\big)^2$ the following asymptotics:

$$d_l\big(\frac{t}{\varepsilon}, \big(\frac{\varepsilon}{t}\big)^{1/4}\big) = \varepsilon\gamma_l \ln\frac{t}{\varepsilon} + \varepsilon O\big(\ln\frac{t}{\varepsilon}\big)^{2/3} + O(1),$$

where $\{\gamma_l\}$ are the parameters, depending on $\{\alpha_l^\pm, \varphi(\alpha_l^\pm), \varphi'(\alpha_l^\pm)\}$ by explicit formulas for problem (1a), (2):

$$\gamma_0 = \gamma_{0,a} = \begin{cases} 0, & \text{if } \ L = 0,\\ \frac{1}{\alpha_0^+ - \alpha_0^-}\big(-\frac{2}{\varphi'(\alpha_0^+)}\big), & \text{if } \ L > 0 \ \text{ and } \ \alpha_0^- < \alpha_0^+,\end{cases}$$

$$\gamma_l = \gamma_{l,a} = \frac{1}{\alpha_l^+ - \alpha_l^-}\big(\frac{2}{\varphi'(\alpha_l^-)} - \frac{2}{\varphi'(\alpha_l^+)}\big), \quad l = 1, \ldots, L-1,$$

$$\gamma_L = \gamma_{L,a} = \begin{cases} \frac{1}{\alpha_L^+ - \alpha_L^-}\big(\frac{2}{\varphi'(\alpha_L^-)}\big), & \text{if } \ L > 0 \ \text{ and } \ \alpha_L^- < \alpha_L^+,\\ 0, & \text{if } \ L = 0.\end{cases}$$

For the problem (1b), (2) we have

$$\gamma_l = \gamma_{l,b} = \frac{c_l}{2}\gamma_{l,a}, \quad l = 0, \ldots, L.$$

Comments

- Theorems 2, 3 are improved versions of Theorems 1a,b from Henkin, 2007.

- Theorems 2, 3 generalize the results of Iljin, Oleinik, 1960, Weinberger, 1990, Henkin, Polterovich, 1991, 1999, Henkin, Shananin, Tumanov, 2004, 2005, Engelberg, Schochet, 2006.

- Theorems 2, 3 answer to the important questions about location in the Cauchy problems (1a,b), (2) of viscous shock-waves Gelfand, 1959, Henkin, Polterovich, 1994, 1999, Liu, Matsumura, Nishihara, 1998.

- Theorems 2, 3 imply the new interesting phenomens: if $L > 0$ and $x \in [c_l t - A\sqrt{t}, c_l t + A\sqrt{t}]$, $l \in \{0, \dots, L\}$, then solutions of (1a,b), (2) converge to shifted travelling waves $\tilde{f}_l\big(\frac{1}{\varepsilon}(x - c_l t - \varepsilon\gamma_l \ln t - \varepsilon O(\ln t)^{2/3} + O(1))\big)$, which generally do not satisfy equations (1a,b) and the positions of which on the x-line depend essentially on the (viscosity) parameter $\varepsilon > 0$. These phenomens lead to the appropriate correction of the Gelfand's suggestion that the main term of asymptotic ($t \to \infty$) of $f(x,t)$, satisfying (1a,b), coincides with solutions of (1a,b) for $\varepsilon = +0$ with the same initial conditions.

- Basing on the works (Kolmogorov, Petrovski, Piskunov, 1937, and Mejai, Volpert, 1999), Gasnikov, 2008, 2009, has proved (only under assumption 1) a rough version of Theorem 2 with shift function $d_l(t) = o(t)$ instead of the precise shift function $d_l(t) = \varepsilon\gamma_l \ln t + \varepsilon O(\ln t)^{2/3} + O(1)$.

The proofs of Theorems 2, 3 are variations of the proofs of Theorems 1 a,b from [H1] and combine earlier techniques (maximum and comparison principles, Lyapunov type functions, Poisson-Green kernels for parabolic type equations) together with new ingredients. For the proof of Theorems 2,3 we need, first, the following comparison proposition from [H1], which is an improvement of Theorem 7.5 from [HP4].

Proposition 1.

Under assumption of Theorem 2 and assumption $\varepsilon = 1$, solutions $f(x,t)$ of (1a,b), (2) satisfy the following estimates:

for every $\gamma > 0$ and $b_l > O(\frac{1}{\gamma})$, $l = 0, \dots, L-1$, there exists $t_0 = O(b_l \gamma)$ such that

$$\varphi^{(-1)}\big(\frac{x - \gamma\sqrt{t}}{t}\big) \leq f(x,t) \leq \varphi^{(-1)}\big(\frac{x + \gamma\sqrt{t}}{t}\big), \tag{9}$$

for $x \in [c_l t + b_l\sqrt{t}, c_{l+1} t - b_{l+1}\sqrt{t}]$ and $t \geq t_0$, where constants c_l are defined by (5a,b).

We need next a proposition, which improves the results of [HST] and of [H1], concerning a priori estimates of derivatives of solutions of the equations (1a,b).

Proposition 2.

Under assumptions of Theorem 2 and of Proposition 1, let $\varepsilon = 1$, $L > 0$, $\varphi(\alpha_l^+) = c_l$, $l = 0, \dots, L-1$, $\varphi(\alpha_l^-) = c_l$, $l = 1, \dots, L$. Let $\tilde{b}_l > b_l > O\big(\frac{1}{\gamma}\big)$, $l = 0, \dots, L$, $\gamma > 0$.

Then the difference $\Delta f = f(x,t) - f(x-1,t)$ for solution of (1b), (2) and the derivative $\frac{\partial f}{\partial x}(x,t)$ for solution of (1a), (2) satisfy the following estimates:

$$\left\{ \begin{array}{c} \Delta f \\ \frac{\partial f}{\partial x} \end{array} \right\} = \frac{1}{\varphi'(\alpha_l^+)(t+1)} + O\big(\frac{\gamma}{\varphi'(\alpha_l^+)(t+1)}\big),$$

for $x \in [c_l t + b_l\sqrt{t}, c_l t + \tilde{b}_l\sqrt{t}]$, $l = 0, \dots, L-1$, $t \geq t_0$, and

$$\left\{ \begin{array}{c} \Delta f \\ \frac{\partial f}{\partial x} \end{array} \right\} = \frac{1}{\varphi'(\alpha_l^-)(t+1)} + O\big(\frac{\gamma}{\varphi'(\alpha_l^-)(t+1)}\big),$$

for $x \in [c_l t - \tilde{b}_l\sqrt{t}, c_l t - b_l\sqrt{t}]$, $l = 1, \dots, L$, $t \geq t_0$, where $t_0 = O\big(\frac{\tilde{b}_l}{b_l} + b_l\gamma\big)$.

The proof of Proposition 2 for solutions of (1b), (2), given in Section 4, develops and corrects the proofs of the corresponding statements in [HST] and [H1].

Propositions 1, 2 imply the following improved version of lemmas 11, 12 from [H1].

Proposition 3.

Under assumptions of Theorems 1, 2 for any $A > 0$, $\theta \in [0, 1)$ and $l \in \{0, \dots, L\}$ the following estimates for shift functions $d_l(t, A)$, defined by (8a,b), are valid

$$|\frac{\partial}{\partial t} d_l(t, A)| = \frac{\gamma_{l,a}}{(t+1)} + O\big(\frac{1}{A(t+1)}\big), \tag{10a}$$

$$|\int\limits_{t-\theta}^{t+1-\theta} \frac{\partial}{\partial \tau} d_l(\tau, A) d\tau| = \frac{\gamma_{l,b}}{(t+1)} + O\big(\frac{1}{A(t+1)}\big), \tag{10b}$$

$$|\frac{\partial}{\partial A} d_l(t, A)| = O\big(A + \frac{1}{A}\big), \quad t \geq t_0, \tag{11}$$

where $\gamma_{l,a}$, $\gamma_{l,b}$ the parameters, defined in Theorem 3.

The following Proposition 4, precised version of Proposition 6 from [H1], gives the main element of the proofs of Theorems 2, 3.

Proposition 4.

Let f be solution of the Cauchy problem (1a), (2) or (1b), (2), where $\varepsilon > 0$. Then under assumptions of Theorem 2 and Proposition 1 we have convergence of $f(x,t)$ to the shifted travelling waves $\tilde{f}_l\big(\frac{1}{\varepsilon}(x - c_l t - d_l\big(\frac{t}{\varepsilon}, \sqrt{\delta}\big))\big)$, $l = 0, \dots, L$, on the intervals $\{x \in \mathbb{R} : \ |x - c_l t| \leq \sqrt{\delta t \varepsilon}\}$ with the estimates

$$\sup_{\{x:\ |x - c_l t| \leq \sqrt{\delta t \varepsilon}\}} |f_l(x,t) - \tilde{f}_l\big(\frac{1}{\varepsilon}(x - c_l t - d_l\big(\frac{t}{\varepsilon}, \sqrt{\delta}\big))\big)| = O(\sqrt{\delta}), \ \text{ if } \ t \geq t_0 = \varepsilon O\big(\ln\frac{1}{\delta}\big)^2.$$

Idea of the proof of Proposition 4 for solution $f(k,t)$ of equation (1b), (2) with $\varepsilon = 1$ and $k \in \mathbb{Z}$ consists (see [H1]) in proving of the following estimate : $\forall \delta > 0$ and $\forall\ l \in \{0, 1, \dots, L\}$

$$\overline{\lim}_{t\to\infty} \sup_{\{n:\ |n - c_l t| \leq \sqrt{\delta t}\}} \Big| \sum_{[c_l t - \sqrt{\delta t}]}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_l(k - c_l t - d_l(t, \sqrt{\delta})))\Big| \leq O(\sqrt{\delta}). \tag{12}$$

The proof of (12) uses nonlinear parabolic type equations for the functions

$$\Delta_l(n, t, d_l(\tau, \sqrt{\delta})) = \sum_{k=[c_l\tau - \sqrt{\delta\tau}]}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_l(n - c_l t - d_l(\tau, \sqrt{\delta})))$$

of variables n, t in the domain $n \in [c_l\tau - \sqrt{\delta\tau}, c_l t + \sqrt{\delta t}] \cap \mathbb{Z}$, $t \in (\tau, \tau + \sqrt{\delta\tau})$. Localized conservation laws are used in order to have a priori boundary estimates:

$$|\Delta_l([c_l t + \sqrt{\delta t}], t, d_l(\tau, \sqrt{\delta}))| = O(1/\sqrt{\tau}).$$

The estimate (12) implies uniform convergence $f(n,t) \Rightarrow \tilde{f}_l(n-c_lt-d_l(t,o(1)))$ in intervals $|n-c_lt| \le o(\sqrt{t})$, $l=0,\dots,L$.

3. Proofs of Theorems 1, 2, 3.

Proof of Theorem 3.

Let $0<A_0<1<A$, $\varepsilon=1$, $l=0,\dots,L$. Estimates (10a,b) from Proposition 3 imply the following

$$d_l(t,A) = \gamma_l \ln\frac{t}{t_0} + O\big(\frac{1}{A}\ln\frac{t}{t_0}\big) + d_l(t_0,A). \tag{13}$$

Estimate (11) from Proposition 3 implies

$$d_l(t,A_0) = d_l(t,A) - O_t(A^2) - O_t\big(\ln\frac{1}{A_0}\big), \quad \text{where} \;\; t \ge t_0. \tag{14}$$

From estimates (13), (14) we deduce

$$d_l(t,A_0) = d_l(t_0,A) + \gamma_l \ln\frac{t}{t_0} + O\big(\frac{1}{A}\ln\frac{t}{t_0}\big) - O_t(A^2) - O_t\big(\ln\frac{1}{A_0}\big).$$

Putting in this estimate $A = \big(\ln\frac{t}{t_0}\big)^{1/3}$ we obtain

$$d_l(t,A_0) = \gamma_l \ln\frac{t}{t_0} + O_t\Big(\big(\ln\frac{t}{t_0}\big)^{2/3}\Big) + d_l(t_0,A) - O_t\big(\ln\frac{1}{A_0}\big). \tag{15}$$

Putting in (15) $t=t_0$ we obtain

$$d_l(t_0,A_0) = d_l(t_0,A) - O_{t_0}\big(\ln\frac{1}{A_0}\big).$$

Let us make the rescalling of equation (15) with parameters $\varepsilon=1$, $\tilde{t}$, $\tilde{x}$, $\tilde{d}$, $\tilde{A}$, $\tilde{t}_0$ into equation with parameters $\varepsilon>0$, t, x, d, A, t_0, using relations

$$\tilde{t}=\frac{t}{\varepsilon},\; \tilde{x}=\frac{x}{\varepsilon},\; \tilde{d}=\frac{d}{\varepsilon},\; \tilde{A}=\frac{A}{\sqrt{\varepsilon}},\; \tilde{t}_0=\frac{t_0}{\varepsilon}=1.$$

We obtain

$$d_l\big(\frac{t}{\varepsilon},A_0\big) = d_l(1,A_0) + \varepsilon\gamma\ln\frac{t}{\varepsilon} + \varepsilon\big(\ln\frac{t}{\varepsilon}\big)^{2/3} - O\big(\ln\frac{1}{A_0}\big). \tag{16}$$

From Proposition 4 we deduce the following estimate

$$\sup_{\{x:\;|x-c_lt|\le\sqrt{\varepsilon}(\varepsilon t)^{1/4}\}} \Big|\tilde{f}_l\Big(\frac{1}{\varepsilon}\big(x-c_lt-d_l\big(\frac{t}{\varepsilon},\big(\frac{\varepsilon}{t}\big)^{1/4}\big)\big)\Big) - \tilde{f}_l\Big(\frac{1}{\varepsilon}\big(x-c_lt-d_l\big(\frac{t}{\varepsilon},A_0\big)\big)\Big)\Big| =$$
$$O(A_0), \quad \text{if} \;\; t \ge t_0 \ge \varepsilon O\big(\ln\frac{1}{A_0}\big)^2.$$

Last estimate implies

$$d_l\big(\frac{t}{\varepsilon}, A_0\big) = d_l\big(\frac{t}{\varepsilon}, \big(\frac{\varepsilon}{t}\big)^{1/4}\big) + O(A_0), \quad \text{if} \quad t \geq \varepsilon O\big(\ln\frac{1}{A_0}\big)^2. \tag{17}$$

From estimates (16), (17) we obtain

$$\begin{aligned} &d_l\big(\frac{t}{\varepsilon}, \big(\frac{\varepsilon}{t}\big)^{1/4}\big) = d_l(1, A_0) + \varepsilon\gamma \ln\frac{t}{\varepsilon} + \varepsilon\big(\ln\frac{t}{\varepsilon}\big)^{2/3} + O(A_0) + O\big(\ln\frac{1}{A_0}\big), \\ &\text{if} \quad t \geq t_0 \geq \varepsilon O\big(\ln\frac{1}{A_0}\big)^2. \end{aligned} \tag{18}$$

This gives statement of Theorem 3 with

$$O(1) = d_l(1, A_0) + 0\big(\ln\frac{1}{A_0}\big).$$

Proof of Theorem 2.

Let $\varepsilon = 1$. Let f be solution of (1a), (2) or (1b), (2) and $\{\tilde{f}_l\}$ be travelling waves solutions for (1a) or (1b) from Proposition 1. Then Proposition 6 from [H1] (i.e. Proposition 4 above with $\varepsilon = 1$) implies $\forall\delta > 0$ estimate

$$\sup_l \sup_{\{x:\ |x - c_l t| \leq \sqrt{\delta} t\}} |f(x,t) - \tilde{f}_l\big(x - c_l t - d_l\big(t, \sqrt{\delta}\big)\big)| = O(\sqrt{\delta}), \quad \text{if} \quad t \geq t_0(\delta). \tag{19}$$

Crucial inequality (6.12) in [H1] shows that one can take in (19)

$$t_0(\delta) = 0\big(\ln\frac{1}{\delta}\big)^2. \tag{20}$$

Results of [W] and [HP4] imply estimates

$$\begin{aligned} &|f(x,t) - \varphi^{(-1)}\big(\frac{x}{t}\big)| = O\big(\frac{1}{\sqrt{\delta t}}\big), \quad \text{if} \quad c_l t + \sqrt{\delta t} \leq x \leq c_{l+1} t - \sqrt{\delta t}, \quad l = 0, \ldots, L-1, \\ &\text{and} \quad |f(x,t) - \alpha_\pm| = O\big(\frac{1}{\sqrt{\delta t}}\big), \quad \text{if} \quad x \leq c_l t - \sqrt{\delta t} \quad \text{or} \quad x \geq c_L t + \sqrt{\delta t}. \end{aligned} \tag{21}$$

Making the rescalling $\tilde{t} = \frac{t}{\varepsilon}$, $\tilde{x} = \frac{x}{\varepsilon}$, $\tilde{d} = \frac{d}{\varepsilon}$, and $\delta = \sqrt{\frac{\varepsilon}{t}}$ into (19), (20), (21) we obtain statement of Theorem 2, if

$$\frac{t_0}{\varepsilon} \geq \big(\ln\, t_0 + \ln\frac{1}{\varepsilon}\big)^2, \quad \text{i.e. if} \quad t_0 \geq \textit{constant}.$$

Proof of Theorem 1.

Let $f^0(x,t)$ be solution of the Cauchy problem (1a), (2) with $\varepsilon = +0$. Let $\delta > 0$, $\varepsilon_k = \delta e^{-k}$ and $t_k = k$, $k = 0, 1, 2, ..$ Let us define continuous function $f(x,t)$ for $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ by the following procedure. For $t \in [t_k, t_{k+1}]$ we put $f(x,t) = f_k(x,t)$ such that

$$\begin{aligned} & f_0(x,0) = f^0(x,0) = f^0(x) \quad \text{and} \\ & \frac{\partial f_k}{\partial t} + \varphi(f_k)\frac{\partial f_k}{\partial x} = \varepsilon_k \frac{\partial^2 f_k}{\partial x^2}, \quad \text{if} \quad t \in [t_{k-1}, t_k], \quad k+1, 2, .. \end{aligned} \tag{22}$$

Applying inductively (22) and result of Kuznetsov, 1975, we obtain for $t \leq t_k$ the estimate

$$\int\limits_{x\in\mathbb{R}} |f(x,t) - f^0(x,t)| \leq O(TVf^0)(\sqrt{\varepsilon_1 t_1} + \sqrt{\varepsilon_2 t_2} + \ldots + \sqrt{\varepsilon_k t_k}), \tag{23}$$

where TVf^0 means a total variation of $f^0(x)$ on $\mathbb{R}$. Substitution of $\varepsilon_k = \delta e^{-k}$ and $t_k = k$ in (23) imply estimate

$$\int\limits_{x\in\mathbb{R}} |f(x,t) - f^0(x,t)| = O(\sqrt{\delta})TV(f^0), \quad t \in \mathbb{R}^+. \tag{24}$$

Using equations (22) and Theorem 2 we obtain for $t \in [t_{k-1}, t_k]$ and
$x \in [c_l t - \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}, c_l t + \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}]$ inequalities

$$|f_k(x,t) - \tilde{f}_{k,l}\big(\frac{1}{\varepsilon_k}\big(x - c_l t - d_l\big(\frac{t}{\varepsilon_k}, \big(\frac{\varepsilon_k}{t}\big)^{1/4}\big)\big)\big)| = O\big(\frac{\varepsilon_k}{t}\big)^{1/4}, \quad k = 1, 2, ..., \quad l = 0, 1, \ldots, L. \tag{25}$$

If $x \in [c_l t + \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}, c_l t - \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}]$ we obtain inequality

$$|f(x,t) - \varphi^{(-1)}\big(\frac{x}{t}\big)| = O\big(\frac{\varepsilon_k}{t}\big)^{1/4}, \quad l = 0, 1, \ldots, L-1. \tag{26}$$

We have also inequalities

$$\begin{aligned} & |f(x,t) - \alpha^-| = O\big(\frac{\varepsilon_k}{t}\big)^{1/4} \quad \text{and} \quad |f(x,t) - \alpha^+| = O\big(\frac{\varepsilon_k}{t}\big)^{1/4}, \quad \text{if} \\ & x \leq c_l t - \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4} \quad \text{or} \quad x \geq c_l t + \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}. \end{aligned}$$

From Theorem 3 (more precisely from equality (18) implying Theorem 3) we have inequalities

$$d_l\big(\frac{t}{\varepsilon}, \big(\frac{\varepsilon_k}{t}\big)^{1/4}\big) = O\big(\varepsilon_k \ln \frac{t}{\varepsilon_k}\big) + O(A_0) + O\big(\ln \frac{1}{A_0}\big), \quad \text{if} \quad t \geq t_0 \geq \varepsilon O\big(\ln \frac{1}{A_0}\big)^2. \tag{27}$$

From (24)-(27) we deduce that $\forall \delta > 0$, $t \in [k-1, k]$, $l = 0, 1, \ldots, L-1$, the following asymptotic equality holds

$$\begin{aligned} & f^0(x,t) = O(\sqrt{\delta}) + O\big(\frac{\varepsilon_k}{t}\big)^{1/4} + \tilde{f}_l\big(\frac{1}{\varepsilon_k}\big(x - c_l t - O\big(\varepsilon_k \ln \frac{t}{\varepsilon_k}\big) - O(A_0) - O\big(\ln \frac{1}{A_0}\big)\big)\big), \quad \text{if} \\ & x \in [c_l t - \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}, c_l t + \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}], \\ & f^0(x,t) = \varphi^{(-1)}\big(\frac{x}{t}\big), \quad \text{if} \quad x \in [c_l t - \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}, c_{l+1} t + \sqrt{\varepsilon_k}(\varepsilon_k t)^{1/4}]. \end{aligned} \tag{28}$$

From (28) we obtain the existence of constants d_l^* such that

$$f^0(x,t) \overset{L^1(\mathbb{R}),t\to\infty}{\Longrightarrow} \begin{cases} \alpha^-, & \text{if } x < c_0t + d_0^* \\ \varphi^{(-1)}(x/t), & \text{if } c_lt + d_l^* \leq x < c_{l+1}t + d_{l+1}^*,\ l = 0,1,\ldots,L-1 \\ \alpha^+, & \text{if } x \geq c_Lt + d_L^*. \end{cases}$$

Remark.

We do not use in our proof apriori condition that $f^0(x,t)$ is a piecewise smooth solution of (1), (2) with $\varepsilon = 0$ like in [P], [KP]. But results of [P], [KP] have advantage, giving explicit formulas for constants $\{d_l^*\}$.

4. Proof of Proposition 2.

We give here the proof of Proposition 2 precising and correcting arguments of [HST] and [H1]. The main tool is as in [HST], [H1] the Green-Poisson type formula associated with operator $u_t' + \Delta u$.

Lemma 1 ([HS],p.1475).

Let $u(x,t)$ be a function defined in the domain $\Omega = \{(x,t):\ t > 0, b < \bar{x} \overset{\text{def}}{=} \frac{x-t}{\sqrt{t}} < \tilde{b} + \sigma\sqrt{t}\}$, $\sigma > \sigma_0 > 0$. Let $\chi(x,t) = \chi_0\big(\frac{x-t}{\sqrt{t}}\big)$, where χ_0 be a smooth function such that

$$0 \leq \chi_0 \leq 1,\quad \chi_0\big|_{(-\infty,b)} \equiv 0,\quad \chi_0\big|_{(\frac{\tilde{b}+b}{2},\infty)} \equiv 1,$$

$$|\chi_0'| \leq \frac{A_0}{\delta},\quad |\chi_0''| \leq \frac{A_0}{\delta^2},\quad \text{where } \delta = (\tilde{b} - b) > 0.$$

Let $\tilde{u}(x,t) = u(x,t)\cdot\chi(x,t)$. Then

$$\tilde{u}(x,t) = \int\limits_{-\infty}^{\infty} G(x-\xi, t-\alpha t)\tilde{u}(\xi,\alpha t)d\xi + \int\limits_{\alpha t}^{t} d\tau \int\limits_{-\infty}^{\infty} G(x-\xi,t-\tau)(\tilde{u}_\tau' + \Delta\tilde{u})(\xi,\tau)d\xi,$$

$$\text{where}\ \ \alpha t > t_0,\ \alpha \in \big(\frac{1+\sigma_0}{1+\sigma}, 1\big),$$

$$G(x,t) = \sum_{n=-\infty}^{\infty} G_n(t)\delta(n-x),\quad \delta(\cdot)\ \text{ the Dirac function,}$$

$$G_n(t) = \frac{t^n}{n!}e^{-t},\quad n \geq 0,\quad G_n(t) = 0,\quad n < 0\ \text{ the Poisson distribution.} \tag{29}$$

Lemma 2 (corrected lemma 4 from [H1]).

Under assumptions of Propositions 1, 2 and Lemma 1 put $u = f - \varphi^{(-1)}\big(\frac{x}{t}\big)$, where f solution of (1b), (2). Suppose that for some $l \in \{0,\ldots,L-1\}$ we have $\varphi(\alpha_l^+) = c_l = 1$, $\tilde{b}_l = \tilde{b} > b_l = b \geq O(\frac{1}{\gamma})$. Then function $\tilde{u}(\xi,\tau) = u(\xi,\tau)\cdot\chi(\xi,\tau)$ satisfies relation

$$\begin{aligned} &\tilde{u}_\tau' + \Delta\tilde{u} = -\frac{\xi-\tau}{\tau}\Delta u\cdot\chi - \frac{1}{2}\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{\xi}{\tau}\big)\big)(\Delta u^2)\cdot\chi - \frac{u}{\tau}\cdot\chi + \\ &\Delta u\cdot\Delta\chi + u(\chi_\tau' + \Delta\chi) + O\big(\frac{\gamma^2}{\tau^2}\big),\quad \tau \geq \tau_0. \end{aligned} \tag{30}$$

Lemma 3 (corrected lemma 5 from [H1]).

Under assumptions of Lemmas 1,2 we have the following representation formula for $\Delta u(x,t)$, if

$$(x,t) \in \Omega = \{(x,t): \ t^* = \alpha t > t_0, x \geq t + \frac{1}{2}(b+\tilde{b})\sqrt{t}\},$$
$$\Delta u = \sum_{k=1}^{5} I_k u, \quad \text{where} \tag{31}$$

$$I_0 u = -\int\limits_{\alpha t}^{t} d\tau \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-\tau) \frac{\xi - \tau}{\tau} \Delta_\xi u(\xi,\tau) \chi(\xi,\tau) d\xi,$$

$$I_1 u = -\int\limits_{\alpha t}^{t} d\tau \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-\tau) \cdot \frac{1}{2} \dot{\varphi}\big(\varphi^{(-1)}\big(\frac{\xi}{\tau}\big)\big) \Delta_\xi u^2(\xi,\tau) \cdot \chi(\xi,\tau) d\xi,$$

$$I_2 u = \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-t^*) u(\xi,t^*) \chi(\xi,t^*) d\xi,$$

$$I_3 u = \int\limits_{\alpha t}^{t} d\tau \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-\tau)(u\chi'_\tau + u \cdot \Delta\chi)(\xi,\tau) d\xi,$$

$$I_4 u = -\int\limits_{\alpha t}^{t} d\tau \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-\tau) \frac{u(\xi,\tau)}{\tau} \chi(\xi,\tau) d\xi,$$

$$I_5 u = -\int\limits_{\alpha t}^{t} d\tau \int\limits_{\bar{\xi} \geq b} \Delta_x G(x-\xi, t-\tau)[\Delta_\xi u(\xi,\tau) \cdot \Delta_\xi \chi(\xi,\tau) + O\big(\frac{1}{\tau^2}\big)] d\xi,$$

where $\bar{\xi} = \frac{t-\tau}{\sqrt{\tau}}$.

Proof.

Putting formula (30) into (29) and applying Δ_x to the left- and right-hand sides of (29), we obtain (31).

Lemma 4 ([HST] lemma 6(iv)+ [H1] lemma 6).

Under assumptions of Lemmas 2,3 we have

$$\int\limits_{\xi} |\Delta_\xi G(x-\xi, t-\tau)| d\xi = \min\{2, O\big(\frac{1}{\sqrt{t-\tau}}\big)\}, \tag{32}$$

$$\int\limits_{\alpha t}^{t} d\tau \int\limits_{\xi} |\Delta_\xi G(x-\xi, t-\tau)| d\xi = \sqrt{1-\alpha}\, O(\sqrt{t}) + O(1). \tag{33}$$

Proof.

Statement (32) is exactly Lemma 6(iv) from [HST]. Statement (33) follows from (32) by the following way

$$\int\limits_{\alpha t}^{t} d\tau \int\limits_{\xi} |\Delta_\xi G(x-\xi,t-\tau)|d\xi =$$

$$\int\limits_{\alpha t}^{t-1} d\tau \int\limits_{\xi} |\Delta_\xi G(x-\xi,t-\tau)|d\xi + \int\limits_{t-1}^{t} d\tau |\Delta_\xi G(x-\xi,t-\tau)|d\xi \leq$$

$$\int\limits_{\alpha t}^{t-1} d\tau O\big(\frac{1}{\sqrt{t-\tau}}\big) + 2\int\limits_{t-1}^{t} d\tau \leq 2 - O(\sqrt{t-\tau})\Big|_{\alpha=t}^{t-1} = O(1) + \sqrt{(1-\alpha)}O(\sqrt{t}).$$

Lemma 5.

Let under assumptions of Lemmas 2,3 we have $\frac{1}{\gamma} \leq b_1 < \tilde{b}_1 < \tilde{b}_2 < b_2$, $\chi = \chi_0\big(\frac{x-t}{\sqrt{t}}\big)$, $\chi_0 = 1$, on $[\tilde{b}_1, \tilde{b}_2]$, $supp\,\chi_0 \subset [b_1, b_2]$, $\delta = \min\{\tilde{b}_1 - b_1, b_2 - \tilde{b}_2\}$, $\Delta_x \chi = O\big(\frac{1}{\delta\sqrt{t}}\big)$, $\alpha < 1$. Then

$$|J_2 u| = O\Big(\frac{\gamma}{\sqrt{1-\alpha}\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)\cdot(t+1)}\Big),$$

$$|J_3 u| = O\Big(\frac{\gamma\tilde{b}_2\sqrt{1-\alpha}}{\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)\cdot(t+1)}\Big)\Big(\frac{1}{\delta^2}+\frac{\tilde{b}_2}{\delta}\Big),$$

$$|J_4 u| = \sqrt{1-\alpha}\,O\big(\sup\frac{|u(\xi,\tau)|}{\tau}\big)\cdot\sqrt{t+1} = O\big(\frac{\gamma}{t+1}\big)\sqrt{1-\alpha}, \quad \text{where}$$

$$\bar{\xi} \geq b_1, \quad \alpha t < \tau < t, \quad t \geq t_0,$$

$$|J_5 u| = \sqrt{1-\alpha}\,O\big(\frac{1}{\delta}\big)\|\Delta u\| + O\big(\frac{1}{(t+1)^{3/2}}\big), \quad t \geq t_0,$$

$$|J_0 u| \leq \sqrt{1-\alpha}\sup_{\bar{\xi}\geq b}\Big|\frac{\xi-\tau}{\tau}\Big|\cdot O(\sqrt{t})\chi(\xi,\tau)\cdot|\Delta u(\xi,\tau)| \leq$$

$$\sqrt{1-\alpha}\,b_2 O\big(\frac{\sqrt{t}}{\sqrt{\alpha t}}\big)\|\Delta u\| \leq \frac{\sqrt{1-\alpha}}{\alpha}b_2\|\Delta u\|,$$

$$|J_1 u| \leq \sqrt{1-\alpha}\sup_{\xi\in[b_1,b_2]}\big|\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{\xi}{\tau}\big)\big)\big|\cdot|\Delta_\xi u^2(\xi,\tau)|\cdot O(\sqrt{t}) \leq$$

$$\sqrt{1-\alpha}\,O\big(\frac{\gamma}{\sqrt{\alpha t}}\cdot\sqrt{t}\big)\|\Delta u\|.$$

Proof.

Estimate of $J_2 u$ follows from Lemma 7 of [HST]. Estimate of $J_3 u$ follows from Lemma 7 of [HST] and Lemma 4. Estimate of $J_4 u$ follows from Lemma 2 and Lemma 4. Estimate of $J_5 u$ follows from Lemma 4 and estimate $|\Delta_\xi\chi(\xi,\tau)| = O\big(\frac{1}{\sqrt{\tau}\delta}\big)$. Estimate of $J_0 u$ follows from Lemma 4 and inequality $|\bar{\xi}| = |\frac{\xi-\tau}{\sqrt{\tau}}| \le b_2$. Estimate of $J_1 u$ follows from Lemma 2 and Lemma 4.

Lemma 6 (corrected and improved Lemma 8 in [H1]).
Under assumptions of Lemmas 2,3 we have

$$|\Delta u| = O\big(\frac{\gamma}{\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)\cdot(t+1)}\big),$$

where $x \in [t + b\sqrt{t}, t + \tilde{b}\sqrt{t}]$, $t \ge t_0$.

Proof.
From (31) we obtain

$$\Delta u - I_0 u - I_1 u - I_5 u = I_2 u + I_3 u + I_4 u.$$

From Lemma 5 we obtain equality

$$\Delta u - I_0 u - I_1 u - I_5 u = \Delta u[1 - \sqrt{1-\alpha}(b_2 + \gamma + O\big(\frac{1}{\delta}\big)] + O\big(\frac{1}{(t+1)^{3/2}}\big) =$$
$$I_2 u + I_3 u + I_4 u = O\big(\frac{\gamma}{t+1}\big)\big[\frac{1}{\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)\cdot\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}\tilde{b}_2}{\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)} \times$$
$$\big(\frac{1}{\delta^2} + \frac{\tilde{b}_2}{\delta} + \sqrt{1-\alpha}\big)\big].$$

If $\sqrt{1-\alpha}(b_2 + \gamma + O\big(\frac{1}{\delta}\big)) < 1$ and γ is small enough then

$$|\Delta u| \le const(b_2, \delta) O\big(\frac{\gamma}{t+1}\big)\big(\frac{1}{\dot{\varphi}\big(\varphi^{(-1)}\big(\frac{x}{t}\big)\big)}\big).$$

Proof of Proposition 2.
Let $c_l = \varphi(\alpha_l^+) = 1$, $x = c_l t + b_l^* \sqrt{t}$, where $b_l^* \in [b_l, \tilde{b}_l]$.
Formula $f = \varphi^{(-1)}\big(\frac{x}{t}\big) + u$ and Lemma 6 imply

$$\Delta f = \Delta \varphi^{(-1)}\big(\frac{x}{t}\big) + \Delta u = \frac{1}{\dot{\varphi}\big(\alpha_l^+ + O\big(\frac{1}{\sqrt{t}}\big)\big)(t+1)} + \Delta u =$$
$$\frac{1}{\varphi'(\alpha_l^+)(t+1)} + O\big(\frac{\gamma}{\varphi'(\alpha_l^+)(t+1)}\big), \text{ if } \ t \ge t_0.$$

Proposition 2 is proved.

5. Multidimensional Burgers type equations. Systems of conservation laws. Conjectures.

Multidimensional analogues of difference-differential Burgers type equations were proposed in Henkin, Polterovich, 1991 as evolutionary equations for efficiency distributions under several efficiency indicators.

Let m, n are levels of two efficiency parameters, $f_{m,n}$ be the proportion of firms which are at the level (m, n).

Let $F_{m,n} = \sum\limits_{k=1}^{m} \sum\limits_{r=1}^{n} f_{k,r}$ be the corresponding distribution function.

Hypotheses: The transition from state (m, n) can be into one of two higher levels $(m+1, n)$ and $(m, n+1)$. The proportion of firms per unit of time moving from the state (m, n) to the state $(m+1, n)$ is proportional to the share of firms being in the state (m, n) and the share of firms being in the state (m, n) and the proportion coefficient is positive function of the share of more advanced firms according to the first indicator. A similar hypothesis is admitted for the transition from (m, n) to $(m, n+1)$.

These assumptions lead to the following equation:

$$\frac{dF_{m,n}}{dt} = \varphi_1(F_m^{(1)})(F_{m-1,n} - F_{m,n}) + \varphi_2(F_n^{(2)})(F_{m,n-1} - F_{m,n}), \tag{34}$$

where

$$\begin{aligned} &F_m^{(1)} = \sup_n F_{m,n}, \quad F_n^{(2)} = \sup_m F_{m,n}, \\ &F_{0,n}(t) \equiv 0, \quad F_{m,0}(t) \equiv 0, \\ &F_{m,n}(0) = 1, \quad m \geq m_0, \quad n \geq n_0. \end{aligned} \tag{35}$$

Equation (34) with boundary conditions (35) implies equations

$$\begin{aligned} \frac{dF_m^{(1)}}{dt} &= \varphi_1(F_m^{(1)})(F_{m-1}^{(1)} - F_m^{(1)}), \\ \frac{dF_n^{(2)}}{dt} &= \varphi_2(F_n^{(2)})(F_{n-1}^{(2)} - F_n^{(2)}). \end{aligned}$$

Let functions φ_1, φ_2 satisfy assumptions 1-3 of §2. Theorem 2, results of [BP], [Con] and [HP1] motivate the following conjecture.

Conjecture.

If $\{F_{m,n}(t)\}$ satisfy (34), (35), then

$$\sup_{m,n} |F_{m,n}(t) - F_m^{(1)}(t)F_n^{(2)}(t)| \to 0, \quad t \to \infty.$$

This statement was proved in [HP1] under condition $L = 0$ in assumptions of §2 for φ_1, φ_2.

Multidimensional analogues of Burgers differential equations

(Viscous conservation laws in dimension $n \geq 1$).

Let $F(x, t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+^1$, be solution of viscous conservation law in dimension $n \geq 1$:

$$\frac{\partial F}{\partial t} + \sum_{j=1}^{n} \varphi_j(F) \frac{\partial F}{\partial x_j} = \varepsilon \Delta F, \ \varepsilon > 0, \tag{36}$$

where functions $\{\varphi_j\}$ satisfy assumptions 1-3 of §2.

Let

$$F(x,0) = f(x_1) + f^0(x), \quad \text{where}$$
$$f(x_1) = \begin{cases} \alpha^{\pm}, & \text{if } \pm x_1 > \pm x_1^{\pm}, \\ f^0(x), & \text{if } x_1 \in [x_1^-, x_1^+],\ x_1 \in \mathbb{R}, \end{cases} \tag{37}$$

and $f^0(x)$ be bounded function with compact support in $\mathbb{R}^n$. Theorem 2 and results of Bauman, Phillips, 1986, Weinberger, 1990, Goodman, Miller, 1999, and Hoff, Zumbrun, 2000, motivate the following conjecture.

Conjecture.

If function $F(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1_+$, satisfy (36), (37), then $(F(x,t) - \hat{f}_1(x_1,x',t)) \Longrightarrow 0$, $t \to \infty$, where $x = (x_1,\dots,x_n) = (x_1,x') \in \mathbb{R}^n$,

$$\hat{f}_1(x_1,x',t) =$$
$$\begin{cases} \tilde{f}_{l,1}(x_1 - c_l t - o(x',t)), & \text{if } |x_1 - c_l t| < A\sqrt{t},\ l = 0,\dots,L, \\ \varphi_1^{(-1)}(\frac{x_1}{t}), & \text{if } c_l(t) + A\sqrt{t} \le x_1 \le c_{l+1}(t) - A\sqrt{t},\ l = 0,\dots,L-1, \\ \alpha^-, & \text{if } x_1 \le c_0 t - A\sqrt{t}, \\ \alpha^+, & \text{if } x_1 \ge c_L t + A\sqrt{t}, \end{cases}$$

$\{c_l\}$ satisfy (5a), (6a) with function $\varphi = \varphi_1$, $\{\tilde{f}_{l,1}\}$ - travelling wave solutions of (1a) with $\varphi = \varphi_1$ and $\varepsilon = 1$, as in Proposition 1,

$$\sup_{x'} |\frac{o(x',t)}{t}| \to 0, \quad t \to \infty.$$

The fundamental applications of multidimensional Burgers type equations for study of the large-scale structure of the Universe were discovered by Zeldovich, 1970, and Gurbatov, Saichev, Shandarin, 1989, through the analysis of the system

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \varepsilon \Delta \vec{v}, \quad \frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0,$$
$$\vec{v}(x,0) = \vec{v}_0(x), \quad \rho(x,0) = \rho_0(x),$$

where $x \in \Omega \subset \mathbb{R}^3$, $\vec{v}(x,t)$ - is the velocity field, $\rho(x,t)$- is the mass density.

Under conditions that $\varepsilon = +0$, the initial distribution of mass ρ_0 is uniform in convex $\Omega \subset \mathbb{R}^3$ and the actual distribution of mass in Ω is known, this model, combined with technique of Monge-Ampere-Kantorovich optimal mass transportation, was recently applied to the reconstruction of initial velocity $\vec{v}_0(x)$ and, as consequence, to reconstruction of $\vec{v}(x,t)$ $\forall$ $x \in \Omega$, $t \ge 0$ (see [FMMS]).

Systems of Burgers type equations (systems of conservation laws)

The problems of finding the asymptotics ($t \to \infty$) of solutions for systems of Burgers type in one spatial variable have been deeply studied starting from fundamental work of

Riemann, 1860, for system of barotropic gaz dynamics

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=\nu\frac{\partial^2\rho}{\partial x^2},$$
$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial P(\rho)}{\partial x}=\varepsilon\frac{\partial^2 v}{\partial x^2},$$

where mass density $\rho\geq 0$, pressure $P(\rho)$ depends only on ρ, $P'(\rho)\geq 0$, viscosity coefficient $\varepsilon=+0$, diffusion coefficient $\nu=+0$.

In this direction many important results on existence and asymptotic stability of (viscous) shock profiles have been obtained by Lax, 1957, Gelfand, 1959, Glimm, 1965, T.-P.Liu, 1985, Szepessy, Xin, 1993, Howard, Zumbrun, 1998, Yu, 1999, Bianchini, Bressan, 2005...

Results of the type Theorems 1,2,3 above for systems of Burgers type have not been obtained yet, even for system barotropic gaz dynamics. Some interesting conjectures in this direction are formulated in Gelfand, 1959, Maslov 1988, Serre 2004.

Similar questions for systems of difference-differential Burgers type equations have been considered only recently (see Benzoni-Gavage, Huot, Rousset, 2003).

This subject is especially important for Schumpeterian models of economical developments of several interacting industries.

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